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Saltation of Plastic Balls in a 'One-Dimensional' Flume

RICHARD GORDON

Mathematical Research Branch, National Institute for Arthritis and Metabolic Diseases Bethesda, Maryland 20014

JACK B. CARMICHAEL

Environmental Dynamics, Inc., P. O. Box 846, Eugene, Oregon 97401

FRANK J. ISACKSON

Department of Polymer Science and Engineering, University of Massachusetts Amherst, Massachusetts 01002

Abstract. A one-dimensional flume was designed and used for the quantitative measurement of the parameters of saltation and underwater collision for plastic balls transported over an unrippled bed by flowing water. Computer simulation reveals that saltation is maintained by the roughness of the bed. A geometric probability argument based only on the grain size distribution is used to derive the distance that a particle travels before entrapment, which turns out to be approximately proportional to the radius, as it has been previously measured. In the appendix the phenomenological stochastic transport equation is rederived from the basic assumptions.

Let us consider a uniform flow of water over a smooth, unrippled bed of sand, the water moving just slightly faster than the threshold velocity, so that few sand grains are in motion at a given time. Much work has been done on the equilibrium of forces and their fluctuations on a grain poised to move [*Jeffreys*, 1929; *Rubey*, 1938; *White*, 1940; *Nevin*, 1946; *Kalinske*, 1947; *Chepil*, 1945, 1958, 1959, 1961; *Einstein and El-Samni*, 1949; *Ippen and Verma*, 1955; *Yalin*, 1958, 1963; *Bagnold*, 1963; *Raudkivi*, 1967]. A grain moves from the bed when lift and drag forces momentarily overcome gravity and friction. Once the grain is on its way, it proceeds downstream in a sequence of trajectories called saltations [*Gilbert*, 1914; *Bagnold*, 1935, 1936, 1941; *Kalinske*, 1942; *Kawamura*, 1951; *Danel et al.*, 1953; *Yalin*, 1958, 1963]. The grain stops [*Einstein*, 1937, 1950] when it enters a depression or 'hole,' whose definition depends on the preceding forces and the local geometry. Later the grain may either resume movement or be covered by other grains coming to rest [*Polya*, 1937; *Galvin and Fristedt*, 1964].

We will consider in detail what maintains saltation, the mechanism of collision with the bed, and how far a grain travels before it finds a hole. In the appendix we will stand back from these mechanical details, and by means of an analogy to the stochastic theory of chromatography of molecules we will look at saltation as a two-state stochastic process in which the grain alternates between mobile and contact 'phases' [*McQuarrie*, 1963; *Carmichael*, 1968].

EXPERIMENTS WITH A ONE-DIMENSIONAL BED

To follow a particle in the course of a few saltations, it is helpful to restrict its motion to a vertical plane, so that it stays in the focal plane of a camera. For this purpose approximately 2000 soft, white vinyl balls with a radius of 0.332 cm and a density of 1.30 g/cm^3 (Parks no. 278 Schmeisser pellet gun ammunition, Park Plastics Co., Linden, N. J.) were purchased and placed in a flume consisting of two plexiglass sheets 0.63 cm thick and 180 $\times$ 20 cm high kept 0.79 cm apart by meter sticks. When tap water (at approximately 20°C) was flowing through this flume at approximately

170 cm³/sec, loose balls saltated readily over the top stationary balls, which constituted a 'one-dimensional' bed. The surface spheres were occasionally disturbed with a glass rod to obtain a new bed configuration. The turbulence of the incoming water was damped by forcing it to flow around a glass rod before it reached the part of the bed to be photographed. The flume was used in a horizontal position only. The balls were approximately hydraulically equivalent to coarse sands (about 0.1 cm in diameter).

The photography was done with a Nikon F camera with a Vari-Clos closeup lens. A variable strobe provided illumination (generally) at 67 flashes/sec as a single ball was dropped upstream in the flume. The shutter was opened for 1 second when the ball reached the field of the camera. An increased aperture of 5–7 f-stops below normal was required for a reasonable exposure of the moving balls without a gross overexposure of the stationary ones.

The velocity profile was measured by pouring soaked square scraps of paper into the flume and photographing them. The profile is remarkably flat and dips only near the bed (Figure 1).

The terminal velocity of the balls in the flume (v_T = 18.1 cm/sec) was measured by dropping them into the flume with the water still and photographing their descent at 33 flashes/sec.

Some of the balls had small dents in them that can be seen rotating in some of the photographs (Figure 2). In a typical example, the rotational energy was 2 ergs, whereas the translational energy was 90 ergs; thus the dynamic contribution of rotation to the trajectory was probably insignificant and has been ignored in all calculations. (During a collision real sand grains with sharp edges should rotate more rapidly. This mechanism is perhaps significant for the loss of translational energy. On the other hand, the rotation of such a particle, at least under water, should also be more rapidly damped than that of a sphere.)

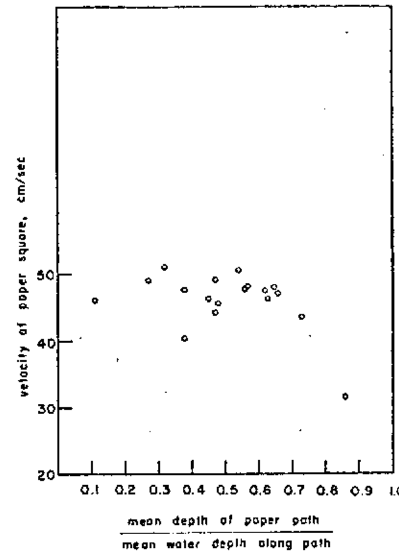

Fig. 1. Vertical velocity profile of the one-dimensional flume measured down from the surface from photographs of the trajectories of paper scraps (see Figure 2). The total water depth varied from 5.1 to 6.5 cm.

NATURE OF THE COLLISION WITH THE BED

A point generally not emphasized in the elementary study of two-body collisions is that, even in an elastic collision, the momentum of the bodies is conserved only if the external forces are small in comparison to the internal impulsive forces developed by the collision [*Halliday and Resnick*, 1962]. This situation was not the case with our plastic balls. In a glancing collision, the motion of the ball was nearly elastic ($\eta \approx 90°$, Figure 3), but in a head-on collision with a ball in the bed, the moving ball seemed not to bounce backward at all. Thus the force due to the flowing water equals or surpasses the internal forces in this case.

In general, when a moving ball struck a stationary ball in the bed, it did not bounce off but rolled around it some distance and left at an angle above the horizontal. (In the rare cases in which the struck ball moved at all, it was totally dislodged.)

Twenty-one simple collisions (in which only one ball in the bed was hit) were analyzed in detail by making tracings of 6.5 × enlargements (Figure 4) onto plastic sheets and taping them on the screen of the 12-inch cathode ray tube attached to an Adage graphics computer. A Fortran program was written that allowed a beam displaying an image of a circle of adjustable radius to be moved around under the control of potentiometers. In this manner the coordinates of the consecutive positions of a

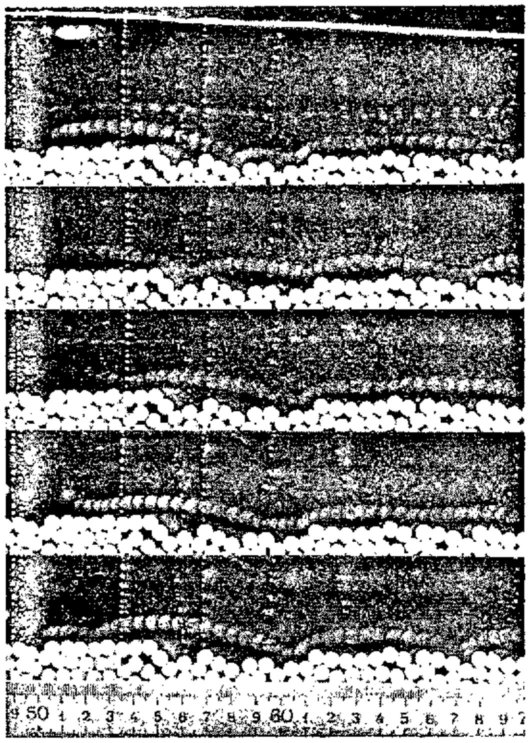

Fig. 2. Trajectories of five balls over the same bed configuration. Each ball was dropped in the flume to the left between the 10- and 30-cm marks. Occasionally a ball effectively rolls at a reduced speed along the bed instead of saltating. The strobe flashed at 67 flashes/sec.

moving ball could be recorded, and the program could calculate the speed, angle from the horizontal, translational energy, and acceleration of the moving ball at each point. Another program took the coordinates of the two images just before the collision occurred and by using a linear extrapolation, calculated the point of contact and, where possible, the initial rolling velocity. The takeoff angle α was measured similarly.

Let β be the angle at which a saltating ball arrives at the bed ball, θ the angle between the horizontal and the line of centers at impact, and α the angle at which the ball takes off (Figure 5). We define $\eta = \theta - \beta$ to be the angle of incidence measured from the line connecting the centers of the spheres at impact. We found a clear correlation between the fractional energy loss and η by measuring the energy just before and just after the collision (Figure 3). (For nearly head-on collisions, where $\eta \approx 0°$, the energy loss is double valued. If the ball stops completely, all energy is lost. However, if the flow manages to accelerate it from a zero velocity, then the net energy loss is much less.)

When a sphere moves in close proximity to a wall, the effective viscosity can increase up to a few hundred times [*Wang,* 1967]. *Eagleson et al.* [1958] find that the linear Stokes relationship holds for particles near a rough bed, even though the Reynolds number is high. Thus we will assume that the combination of these two effects in our flume results in a linear force–velocity relationship, despite Reynolds numbers of up to 10^3. (*Yalin* [1958, 1963] assumes a quadratic relationship.) Let the proportionality constant be s. Then at terminal velocity

$$sv_T = m_b g \tag{1}$$

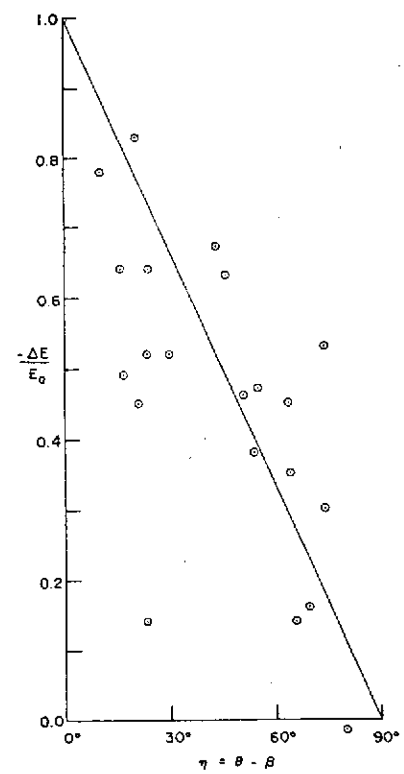

Fig. 3. Relative loss of translational energy immediately after collision of a ball with a stationary ball in the bed versus the angle of incidence η (see Figure 5).

where m_b is the buoyant mass of the ball and g is the gravitational acceleration.

During a collision the moving ball rolls over the ball with which it has collided. Let us assume that the water flow around the struck ball follows the same arc, at least to the top. If v is the speed of the ball, then

$$m\frac{dv}{dt} = s(w - v) - m_b g \cos\theta \qquad (2)$$

where m is its mass and w the water speed. Since $v = 2r\, d\theta/dt$, we obtain

$$\frac{d^2\theta}{dt^2} = \frac{m_b g}{2rm}\left[\frac{(w - 2r\, d\theta/dt)}{v_T} - \cos\theta\right] \qquad (3)$$

This nonlinear second order differential equation must be solved subject to two boundary conditions. At $t = 0$, $\theta(0)$ is given by the contact point. For $d\theta/dt$ at $t = 0$, we assume that (1) the velocity component along the line of centers at initial contact is lost on impact and that (2) the tangential component is completely retained.

The first assumption is based on the fact that no backward bounces were ever observed. In seven of the 21 collisions analyzed in detail, it was possible to calculate the initial rolling speed and to compare it to the tangential component of the velocity just before impact. The ratio of these speeds varied from 0.66 to 1.47, the mean being 1.01; thus the second assumption seems approximately justified.

The rolling ball should separate from the ball in the bed when the centrifugal force balances the force of gravity:

$$mv^2/2r \geq m_b g \sin\theta \qquad (4a)$$

or when

$$(d\theta/dt)^2/\sin\theta \geq m_b g/2rm \qquad (4b)$$

Experimentally, the ratio of the left to the right half of inequality 4a varied at takeoff from 1.1 to 5.3 for 19 cases, the average being 3.38; thus the takeoff angles α were lower than those calculated by this crude theory. The water flow probably has an inward radial component w_r pressing the rolling ball against the stationary one. The separation criterion then becomes

$$\frac{(d\theta/dt)^2}{\sin\theta} \geq \frac{(m_b g + s w_r)}{2rm} \qquad (5)$$

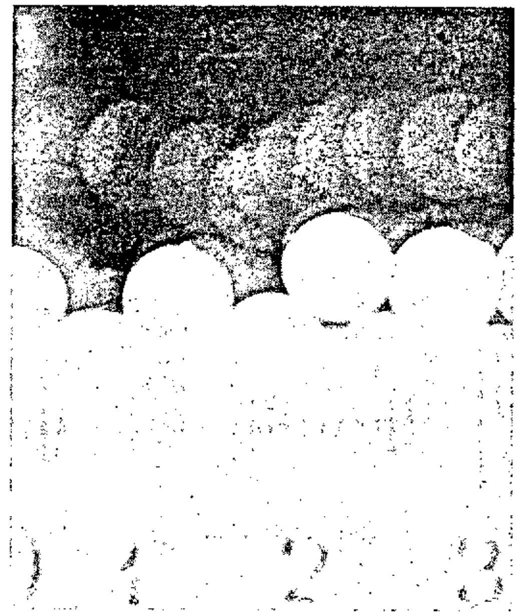

Fig. 4. Close-up of a single collision photographed with a strope at 67 flashes/sec. The halos around the bed balls are their reflections off the back wall of the flume. The ball is traveling to the right.

from which we estimate w_r to be 43 cm/sec. This value is comparable to the flow velocity (Figure 1).

The water flow during a collision is obviously more complicated than we have assumed, and some attempt should probably be made to measure the flow pattern directly. Thus an underwater collision of a saltating grain is fairly complicated, even if that grain is a sphere, and the hydrodynamics of such underwater collisions has yet to be developed.

SALTATION TRAJECTORIES

The spheres bouncing down our one-dimensional flume exhibited typical saltation trajectories, the arc starting off at a relatively steep angle and ending with the ball striking the bed at a low angle (Figures 2 and 6). Each ball dropped into the flume went through a unique set of trajectories. Nevertheless, it is interesting that a given bed configuration has 'hot spots' where collisions occur preferentially (Figure 2). (These spots may be due to downstream wakes.)

Let x represent the distance along a horizontal bed and y the height above the bed. Then the

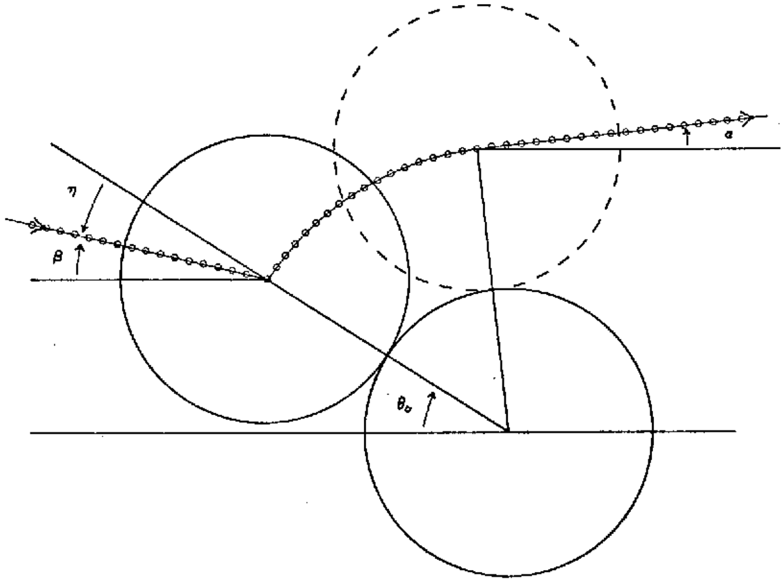

Fig. 5. Definition of angles during a collision. The dotted line indicates the trajectory of the center of the moving ball.

x and y components of the force on a ball moving above the bed are given by

$$F_y = m_b g - s v_y = m\,(dv_y/dt)$$
$$F_x = s[v_x - w(y)] = m\,(dv_x/dt) \qquad (6)$$

where $w(y)$ is the water velocity as a function of height. Let $\tau = m/s = m v_T/(m_b\, g) = 0.080$ second for our plastic balls. The parameter τ is the relaxation time for the fluid drag on the ball. Then

$$\frac{dv_y}{dt} = (-v_y/\tau) - (m_b g/m)$$
$$\frac{dv_x}{dt} = \{w[y(t)] - v_x\}/\tau \qquad (7)$$

The y component can be solved completely [*Fowles*, 1962]:

$$v_y(t) = -v_T + (v_T + v_{y0})e^{-t/\tau}$$
$$y(t) = y_0 - v_T t + \tau(v_T + v_{y0})(1 - e^{-t/\tau}) \qquad (8)$$

where v_{y0} and y_0 are the y velocity and height at time $t = 0$, respectively. The x component is complicated by the dependence of the water speed on height and thus on time [*Coddington*, 1961]:

$$v_x(t) = \frac{e^{-t/\tau}}{\tau}\int_0^t e^{t/\tau} w[y(t)]\, dt + v_{x0}e^{-t/\tau} \qquad (9)$$

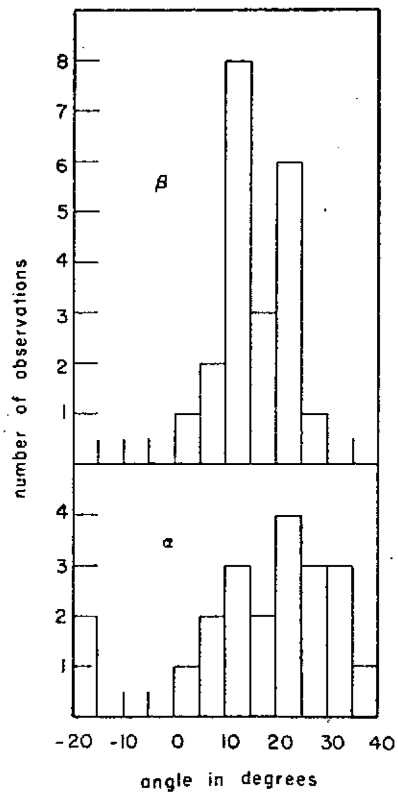

Fig. 6. Distribution of takeoff angles α and collision angles β, both measured from the horizontal. The water speed was 35 cm/sec.

If w is constant,

$$v_x(t) = w + (v_{x0} - w)e^{-t/\tau} \tag{10}$$

$$x(t) = x_0 + wt + \tau(v_{x0} - w)(1 - e^{-t/\tau})$$

The ball reaches the top of its arc at the time t_h given by $v_y(t_h) = 0$, or

$$t_h = -\tau \ln [v_T/(v_T + v_{y0})] \tag{11}$$

and rises to the height

$$y_h = y(t_h) = y_0 + \tau v_{y0} + \tau v_T \ln \left(\frac{v_T}{v_T + v_{y0}}\right) \tag{12}$$

The average value of v_{y0} for all saltations photographed was 7.4 cm/sec, which corresponds to $y_h - y_0 = 0.10$ cm and $t_h = 0.027$ second (cf. Figure 2).

If impact occurs when the ball is at height y_1, the time of impact t_1 is given (equation 8) by the transcendental equation

$$\left(\frac{v_T}{v_T + v_{y0}}\right)\frac{t_1}{\tau} = 1 - e^{-t_1/\tau} + (y_0 - y_1)/[\tau(v_T + v_{y0})] \tag{13}$$

which may be solved for t_1 graphically or numerically by iteration. For $y_1 = y_0$ we have the upper bound

$$t_1 < \tau[1 + (v_{y0}/v_T)] \tag{14}$$

Experimentally, v_{y0} was in the range 1.9 to 12.8 cm/sec; thus a complete saltation trajectory takes less than two relaxation times ($v_T = 18.1$ cm/sec), and at collision the ball has not yet reached terminal velocity. Thus we cannot expect Bagnold's relationship [*Raudkivi*, 1967]

$$\tan \beta = -v_y(t_1)/v_x(t_1) \overset{?}{=} v_T/w \tag{15}$$

to hold for saltation under water. It has been claimed [*Raudkivi*, 1967] that 'if the grain does not rise high enough to reach full wind velocity, it will also not reach the full terminal velocity of fall and the angle β will not be affected much.' However, by subtracting v_T/w in equation 15, we find that v_T/w is an upper bound for $\tan \beta$ whenever $\tan \alpha = (v_{y0}/v_{x0}) > (-v_T/w)$, which holds for all but extreme backward bounces. Indeed, in our experiments when $w = 35$ cm/sec, the measured values of

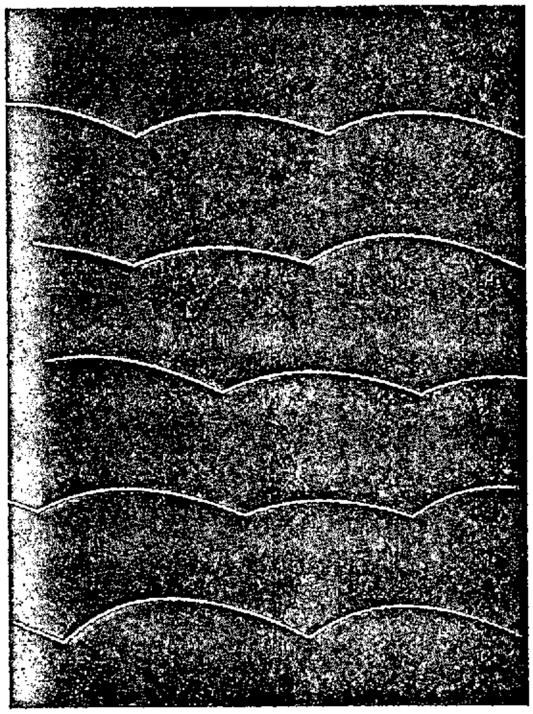

Fig. 7. A set of typical saltation trajectories from the Runge-Kutta simulation. In this case $\tau = 0.04$, and θ was chosen uniformly at random between 0° and 12°. The particle moves from left to right. The water speed w was 40 cm/sec.

β rarely approached the predicted value of 27° (Figure 6).

If dimensionless velocities are defined in relation to the terminal velocity ($\nu_x = v_x/v_T$, $\nu_y = v_y/v_T$, $\omega = w/v_T$) and the relaxation time τ is taken as the time unit, then the dimensionless equations of motion are functions of the single parameter ω:

$$\begin{aligned} d\nu_y/dt &= -\nu_y - 1 \\ d\nu_x/dt &= \omega - \nu_x \end{aligned} \tag{16}$$

Bagnold [1935] suggested that for a given initial speed a sand grain saltates a maximum distance downstream if it takes off vertically. To the contrary, we found from the above equations that at a dimensionless initial speed $\nu = 10$, for instance, the sand grain takes off optimally at angles α of 28°, 45°, 63°, and 84° for $\omega = 0.5$, 1, 2, and 10, respectively. His assumption seems correct only for quite large water speeds or very small initial particle speeds.

MAINTENANCE OF SALTATION

Bouncing is a mechanism whereby translational energy may be transformed from one direction to another. Since a saltating grain arrives at the bed at a low angle β, a collision essentially converts energy in the x direction into energy in the y direction. We have seen that an underwater collision retains some of the character of an elastic bounce through the retention of the tangential velocity component. Nevertheless, a certain fraction of the energy is lost (Figure 3), and if this loss is not completely made up fairly soon by energy taken from the moving water, saltation may damp out.

To investigate the continuation of saltation, we wrote a computer simulation in Fortran by using an idealized bed that the saltating particle always strikes at the same height and for which each point struck has a controllable angle θ (the angle of inclination being $90° - \theta$). We assumed that the normal component of velocity is lost on impact and that the tangential component is retained. Rather than calculate the detailed dynamics of collision rolling, we assumed that no energy is gained during the collision and that the takeoff angle is given by $\alpha = \theta \sin \eta = \theta \sin (\theta - \beta)$. Our units of time and distance were fixed by setting $\tau = 1$ and $v_T = 1$. The velocity profile was taken as constant.

At first we allowed θ to vary randomly with a uniform distribution between 0° and some maximum at each collision. The trajectories were calculated by using a fourth order Runge-Kutta integration of equations 7 [*McCracken and Dorn*, 1964], and the results were displayed as they were computed on the cathode ray tube of an Adage graphics computer (Figure 7). After the parameters were varied for a while to see what would happen, it became clear to us that the roughness of the bed was of primary importance for the continuation of saltation. In particular, if the bed was smooth ($\theta = 90°$), the saltations damped out, and the particle rolled along.

To quantify these results, we ran a series of different simulations with θ fixed and calculated only the starting and final parameters for each arc by using equations 8, 10, and 13. Simulation runs were started at both high and low energy and continued until a 'steady state' trajectory was obtained. The same final arc was obtained in both cases. The parameters of these arcs as a function of θ are shown in Figure 8. Their most remarkable property is that for a given water speed ω there is a critical value of θ below which a ball will not move at all. (Fixing θ in the simulation is equivalent to replacing a statistical variable by its mean.)

DISTANCE BETWEEN ENTRAPMENTS

The stochastic theory of gel permeation chromatography [*Carmichael*, 1968] assumes that the gel beads contain numerous holes in which polymer molecules are temporarily trapped. There is a spectrum of hole sizes, and the basic assumption is that a polymer molecule may be trapped by any hole wider than itself. In a sand bed, the holes are formed by the sand itself, so that we may speak descriptively of the 'self chromatography' of sand. We will attempt to define a hole and to derive the distribution of hole sizes and the mean distance to the next hole.

We ignore saltation temporarily and pretend that a grain rolls over the bed in more or less continuous contact with it. Suppose that a hole looks circular from the top, and let the surface density of holes of radius R be $\rho(R)$. If a grain of radius r were to roll a distance l, then its center would have passed over $2Rl\,\rho(R)$ holes of radius R. Therefore the number of holes per unit path length $h(r)$ for a grain of radius r is

$$h(r) = 2 \int_r^\infty R\rho(R)\, dR \qquad (17)$$

The term $\rho(R)$ depends on the statistics of placement of grains above and below the mean bed level. This placement in turn depends on the packing and the grain radius distribution $g(r)$. In the spirit of geometric probability [*Kendall and Moran*, 1963], consider a randomly close packed bulk sample of spherical particles, and imagine creating a bumpy surface by passing a mathematical plane through such sand and removing all grains whose centers lie above the plane. The interstices may be imagined filled with plaster. (This condition reduces the number of holes for the smaller particles, but they are normally absent from natural sands anyway.) Thus each grain that intersects the reference plane and is removed leaves a dent in that plane, and each grain that remains leaves a bump. By symmetry, the distribution $B(y)$ of the heights of the bumps is identical to the distribution of the depths of the dents. We will assume for now that such a surface is a good approximation of a natural sand bed.

If we assume a bump of height y above the reference plane, the possibility that it is due to a particle of radius $r \geq y$ whose center is below the plane is proportional to $g(r)$. Therefore the distribution of bump heights (and dent depths) is given by

$$B(y) = \left[\int_y^\infty g(r)\,dr\right] \Big/ \int_0^\infty \left[\int_y^\infty g(r)\,dr\right] dy \tag{18}$$

The number of dents equals the number of bumps, and we next assume that their place-

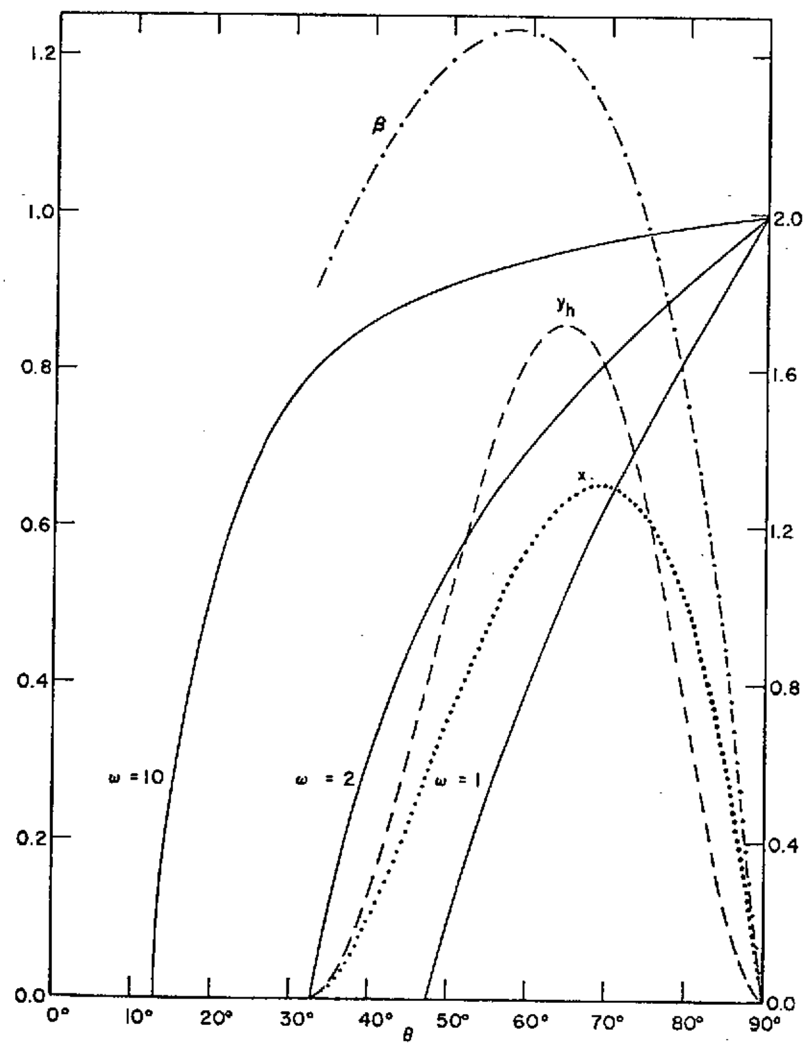

Fig. 8. Properties of saltation arcs over an idealized bed for which each point struck is at a fixed angle θ (see Figure 5). The dimensionless mean x velocity $\overline{v_x}$ divided by the dimensionless water speed ω (terminal velocity $v_T = 1$) is shown (solid lines, left scale). For $\omega = 2$ the distance traveled along the bed x (dotted line, right scale), the maximum height y_h (dashed line, 1/10 left scale), and the angle β (dashed and dotted line, 10 $\times$ left scale, in degrees) are shown. This β is considerably less than Bagnold's $\beta = \tan^{-1}(1/2) = 26.57°$. The angle α (not shown) is greater than β everywhere and follows a curve of a similar shape.

ment on the bed is not correlated. If the bed were one-dimensional (as in our experimental flume), then the probability of a dent being followed by a dent in the downstream direction would equal the probability of the dent being followed by a bump. If the depth of the hole formed by the dent is defined as the depth of the dent in the first case and as the sum of the depth of the dent and the height of the bump in the second case, then the probability $H(z)$ that a hole has a depth z is

$$H(z) = \tfrac{1}{2}[B(z) + B * B(z)] \qquad (19)$$

where the asterisk denotes convolution [*Feller*, 1966]:

$$B * B(z) = \int_0^z B(z - y)B(y)\, dy \qquad (20)$$

A two-dimensional bed could be treated as such, but the combinatorics would be more complicated and the hole harder to define. Even for a one-dimensional bed, other cases are possible, such as a bump followed by a higher bump or two dents in a row followed by a bump, and so forth. We assume that equation 19 yields an adequate approximation to such an infinite series.

We introduce the closeness of the packing of the sand by assuming that the radius of a hole is equal to its depth, so that $\rho(R) = kH(R)$, where k is a constant. If a is the fraction of the bed surface covered by holes of all sizes, then k may be evaluated from

$$a = \int_0^\infty \pi R^2 \rho(R)\, dR = k \int_0^\infty \pi R^2 H(R)\, dR \qquad (21)$$

if a value of a is given. That $a = 1/2$ seems reasonable.

In summary, we start only with the grain size distribution $g(r)$, create a surface with reasonably well-defined statistical properties, evaluate the bumpiness $B(y)$ in terms of $g(r)$ (equation 18), obtain the distribution of hole depths $H(z)$ in terms of $B(y)$ (equation 19), and then obtain the surface density of holes $\rho(R) = kH(R)$, where k is a constant that can be estimated from equation 21. The mean distance between holes for a rolling particle of radius r, $1/h(r)$, is then evaluated from $\rho(R)$ by using equation 17.

Evaluation of this sequence of integrals is tedious even for the simplest form of the distribution $g(r)$. Thus we chose to use numerical integration and to take $g(r)$ as the log normal distribution:

$$g(r) = \frac{\exp\{-[\ln(r/\bar{r})]^2/(2\sigma^2)\}}{r\sigma(2\pi)^{1/2}} \qquad (22)$$

where $\bar{r}$ is the geometric mean radius and σ is the logarithm standard deviation. Natural sands sometimes follow this distribution [*Krumbein*, 1954; *McIntyre*, 1959], and there is even theoretical justification for it [*Kolmogoroff*, 1941; *Aitchison and Brown*, 1957; *Herdan*, 1960].

Einstein [1937, 1950] reported that a sand grain travels an average distance of 200 times its radius before stopping. In Figure 9 we have calculated the number of radii grains travel versus their radius for a few values of σ. This number is approximately constant for most of the particles in a given distribution $g(r)$ and remarkably so for larger σ. Moreover, the number of radii traveled is roughly independent of σ. We have taken *Einstein*'s [1937, Tables 20 and 22] original data for three sizes of grains passed over two bed mixtures to show that the calculated curves are as 'flat' as the range and scatter of his data could require.

However, our curves indicate that the holes are only six to 15 radii apart, instead of 200. The major part of this discrepancy is of course due to saltation, which we have been ignoring in this section. A saltating particle jumping over many holes samples only a small part of the bed during its collision with it. *Bagnold* [1935] has found that the distance traveled by a saltating particle is fairly independent of its size; thus the proportionality of the distance traveled to the radius should remain roughly constant with r but should be increased. In the one-dimensional flume, the average saltation length was 3.5 cm; thus if the ball samples one ball diameter of bed per bounce, the calculated mean distance before entrapment becomes 40 radii. (Because of the short length of our flume, the difficulties of defining a representative bed configuration, and the packing regularities of the bed, we did not attempt to collect data on entrapments. They were occasionally observed, especially with freshly disturbed beds.)

Other factors that could affect the calculated distance to entrapment (the first three tending to increase it) are. (1) Our hypothetical bed would tend to be flattened with use both by filling in holes and by segregation, which violates the assumption that 'particles are equally available at the surface and in the main body of the bed' [*Einstein and Chien,* 1953]. (2) Since one particle can roll sideways around another rather than go over it, a full two-dimensional treatment of holes in a bed may result in a lower density of holes $\rho(R)$. (3) Our definition of a hole does not take into account the angle of repose of the particle in the hole, which depends on the local fluid velocity. Thus some of our holes would not retain all particles that could fit in them. (4) *Einstein's* [1937] data pertain to gravel of radius 0.8–1.7 cm; such pebbles are not hydraulically equivalent to natural sands.

Although there are a number of statistical

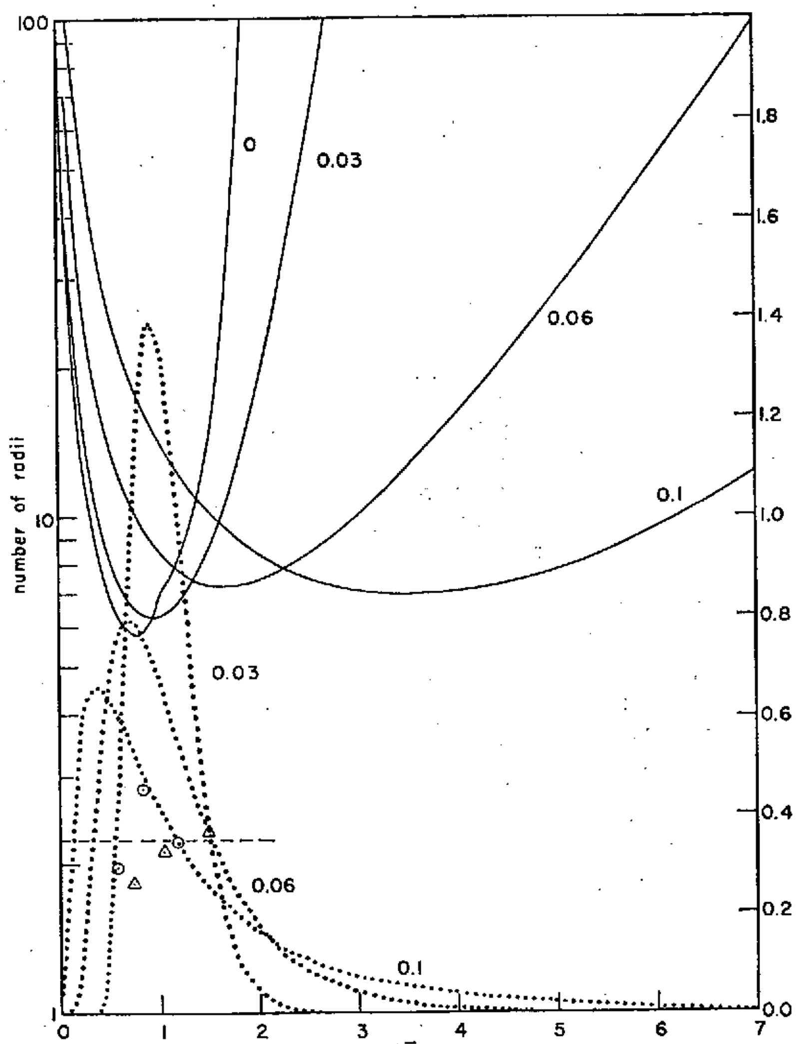

Fig. 9. The number of radii traveled (solid lines, left scale) versus $r/\bar{r}$ for four values of σ. The corresponding log normal size distributions $g(r)$ are shown (dotted lines, right scale). (For $\sigma = 0$, $g(r)$ is a Dirac delta function at $r/\bar{r} = 1$.) The plotted points (100 $\times$ left scale) are *Einstein's* [1937] data for three sizes of gravel (1.7, 2.4, and 3.4 cm in diameter) over two beds (circles are beds 2.9 cm in diameter, and triangles are those 2.3 cm in diameter). His proposed constant relationship, the number of radii being 223 (average of his data), independent of r is shown (dashed line).

results on the random close packing of equal sized spheres [*Coxeter*, 1958, *Scott*, 1960, 1962; *Bernal and Mason*, 1960; *Frisch and Stillinger*, 1963], little work has been done on packings with a distribution of grain sizes [*Krumbein*, 1954; *Kahn*, 1956]. The preceding result on the distribution of holes on a heterogeneous flat bed is the first such theoretical result known to us, except for the derivation of $g(r)$ from sections of packed material [*Kendall and Moran*, 1963]. Many other properties of sand beds, such as the distribution of contact angles θ or the collisional scattering of grains perpendicular to the downstream direction, could perhaps be profitably studied from the viewpoint of geometric probability.

DISCUSSION

Bagnold [1956] claims that 'owing to viscous effects in liquids the grain's velocity on return to the bed is insufficient to cause any observable rebound or any disturbance of the bed grains, at any rate at feeble flow strengths . . . (just above threshold.)' However, our plastic balls bounced readily and occasionally dislodged a ball in the bed. *Danel et al.* [1953] observed for sand in both air and water that 'the particles ricochet off the bed and thus lose only a portion of the kinetic energy imparted by the fluid.'

The importance of saltation as a means of sand transport in water has been questioned [*Kalinske*, 1947] because the height of the trajectories is 1/800 that in air. However, as *Danel et al.* [1953] pointed out, 'rolling on the bed without a clear-cut separation or simple loss of contact is not easy to imagine,' so that almost by definition any sand transport involving single grains occurs by saltation. The major exception is the rolling or skidding of, say, a pebble over much finer sand, where the size ratio is quite large [*Einstein and Chien*, 1953].

Turbulence is necessary to dislodge a grain and start it moving (but cf. *Yalin* [1963]). However, as *Bagnold* [1954, 1966] has pointed out, once a large number of interacting particles are on the move, turbulence need not be invoked to explain their continued motion. But, as we have seen in the section on the maintenance of saltation, all that is really necessary for continued saltation of a particle is that the water velocity be sufficiently high (Figure 8) over a rough bed. The flow can be effectively laminar.

In summary, we have designed, constructed, and used a one-dimensional flume that makes the dynamics of saltating particles readily visible. We found that an underwater collision has an elastic and an inelastic component, that particles do not reach terminal velocity before collision, and that saltation is maintained by the roughness of the bed, which 'transforms' the kinetic energy of the particle from a horizontal to a vertical direction. (A considerable amount of kinematic and hydrodynamic experiment and analysis has yet to be done for real sand grains.) We proceeded to use a rough geometric probability argument to derive the distance a particle goes before entrapment by a 'hole' in the bed, which turned out to be approximately proportional to its radius, in agreement with the experiments of *Einstein* [1937]. In the appendix we rederive the probability that a particle travels a given distance in a given amount of time. Our study was inspired by analogies between the motion of sand and the chromatography of polymer molecules. This field is fruitful for interdisciplinary studies, as the 'soil polymer parallels' of *Krizek* [1968] also show.

APPENDIX: STOCHASTIC ANALYSIS OF SAND TRANSPORT

Other authors (e.g., *Hubble and Sayre* [1964]) have rederived Einstein's equations for the probability distribution of grain movement associated with saltation. This derivation assumes a distribution of distances traveled (which are proportional to times spent) by the particle while it is traveling in the fluid state and out of contact with the bed. This derivation also assumes a distribution of times spent by the particle while it is in contact with the bed. However, no distance is assumed to be traveled by the particle while it is in contact with the bed.

We will now derive the probability distribution (equation A14) allowing a distribution of times spent and distances traveled by the particle while it is both in and out of contact with the bed. Rolling, for example, is permitted under these assumptions. Then Einstein's result is obtained as the aforementioned special case (equation A17).

Consider a model of sand transport in which a sand grain can be in either of two states. State 1 will be called the fluid state. While it is in this state, the sand grain is moving in the water phase and remains out of contact with the bed. State 2 will be called the contact state. While it is in state 2, the sand grain is in contact with the bed of the flume or the natural bed of a flowing body of water. The sand grain can be either stationary or rolling while it is in contact with the bed.

When a sand grain passes from the fluid state to the contact state, this transition will be called the process of capture. When a sand grain moves from its position in contact with the bed into the fluid state, this transition will be called the process of release.

The term saltation has been coined to describe the bouncing motion of sand grains when they are in transport by wind or water. In our nomenclature, therefore, the term saltation describes the entire cycle of a sand grain consisting of capture, time spent in the contact state, release, and time spent in the fluid state.

To complete the framework for the model, we also make the following two physical assumptions: (1) that sand grains move independently of one another and (2) that the processes of capture and release occur independently of one another and that each of these processes is random.

The randomness of the processes of capture and release assumed in assumption 2 can arise physically from random fluctations in the water velocity in the vicinity of the sand grain while it is in either the fluid or the contact state. We define

$$\text{prob (fluid} \rightarrow \text{contact)} = \lambda_1 \, \Delta t + 0(\Delta t) \quad \text{(A1}a\text{)}$$

(read as the probability of the sand grain making the transition from the fluid state to the contact state, i.e., the probability of capture) and

$$\text{prob (contact} \rightarrow \text{fluid)} = \lambda_2 \, \Delta t + 0(\Delta t) \quad \text{(A1}b\text{)}$$

where λ_1 and λ_2 are the rate constants for capture and release, respectively. Each is a first order rate constant with units of $(\text{time})^{-1}$. The magnitudes of these rate constants will in general be functions of the size of the grain being transported, the grain size distribution in the bed, and the mean water velocity near the bed.

Each of equations A1a and A1b defines the transition probability for a Poisson process. That is, we have assumed for each of the processes defined by equation A1 that (1) events occur randomly, (2) the probabilities of events in any two nonoverlapping intervals of time are independent of one another, (3) the probability of an event in an interval of time is not a function of time, and (4) the probability of a total of two or more events occurring in $(t, t + \Delta t)$ is $0(\Delta t)$.

The solution of equations A1a or A1b [*Bharucha-Reid*, 1960] yields

$$p(i; \lambda t) = \frac{(\lambda t)^i}{i!} \exp(-\lambda t) \qquad \text{(A2)}$$

where $p(i; \lambda t)$ represents the probability that the system has undergone i events at time t. As a solution to equation A1a, for example, equation A2 would represent the probability that (given that a sand grain began in the contact state at $t = 0$), an additional $i - 1$ captures had occurred by time t, the total thus being i separate instances when the sand grain resided in the contact state. Equation A2 represents Poisson distributions in time for the times spent by sand grains in the fluid and contact states.

As it stands, $p(i; \lambda t)$ is already normalized in i. It must, however, be normalized in t, since we ultimately seek a normalized probability distribution in time for the sand grains to travel an arbitrary but fixed distance.

Define

$$P_i(t) = \frac{p(i; \lambda t)}{\int_0^\infty p(i; \lambda t) \, dt} = \lambda p(i; \lambda t) \qquad \text{(A3)}$$

where $P_i(t) \, dt$ is the probability that a sand grain undergoes its ith capture or release in the time interval $(t, t + dt)$ and $P_i(t)$ is a probability distribution in time.

Let t_f and t_c be the times spent by sand grains in the fluid and contact states, respectively:

$$P_i(t_f) = \lambda_1 p(i; \lambda_1 t_f) \qquad \text{(A4}a\text{)}$$

$$P_i(t_c) = \lambda_2 p(i; \lambda_2 t_c) \qquad \text{(A4}b\text{)}$$

Assumption 2 required that the random variables associated with the times t_f and t_c must be statistically independent. Thus $t_f + t_c$

is the total time t for a sand grain to travel some fixed but arbitrary distance.

Since the random variables associated with the times t_f and t_c are statistically independent, the probability distribution of the sum of the times t_f plus t_c (t) is the convolution of the probability distributions in the times t_f and t_c:

$$P_i(t_f + t_c) = P_i(t) \equiv P\{T = t \mid i\} \quad (A5a)$$

and

$$P\{T = t \mid i\} = P_i(t_f) * P_i(t_c) \quad (A5b)$$

where the asterisk denotes the convolution operation applied to the two integral-valued probability distributions.

$P\{T = t|i\}$ represents the conditional probability that the random variable T associated with the total time required for the sand grain to move a prescribed distance has a particular value t, on the assumption that the sand grain had undergone i captures in traveling that distance.

Equation A5 is solved by using the probability generating function [*Feller*, 1968] $h(s;\ \lambda t)$ defined as

$$h(s; \lambda t) = \sum_{i=0}^{\infty} p(i; \lambda t) s^i \quad (A6)$$

For the special case of two Poisson distributions,

$$h(s; \lambda_1 t_f + \lambda_2 t_c) = h(s; \lambda_1 t_f) \cdot h(s; \lambda_2 t_c) \quad (A7)$$

The inversion of $h(s;\ \lambda_1\ t_f + \lambda_2\ t_c)$ to yield a probability distribution is straightforward. The solution of equation A7 by using the definition of the generating function from equation A6 yields the Poisson distribution

$$p(i; \lambda_1 t_f + \lambda_2 t_c) \quad (A8)$$

Therefore

$$P\{T = t \mid i\} = \lambda_1 \lambda_2 p(i; \lambda_1 t_f + \lambda_2 t_c) \quad (A9)$$

Define $P(t)$ as the probability as a function of time that the sand grain travels a particular specified distance. Then

$$P(t) = \sum_{i=0}^{\infty} P\{T = t \mid i\} P\{I = i\} \quad (A10)$$

where $P\{I = i\}$ represents the absolute probability that the sand grain has been captured i times:

$$P\{I = i\} = \frac{(\lambda_1 t_f)^{i+1}}{(i+1)!} \exp(-\lambda_1 t_f) \quad (A11)$$

The exponent $i + 1$ appears in equation A11 because, in addition to being captured i times in the course of transport, the sand grain began its series of saltations in the contact state at $t = 0$.

Evaluation of $P(t)$ in equation A10 by using the expression for $P\ \{I = i\}$ from equation A11 and $P\ \{T = t|i\}$ from equation A9 yields

$$P(t) = \exp[t(\gamma\lambda_2 - \lambda_2 - 2\gamma\lambda_1)]$$
$$\cdot \lambda_1\lambda_2[\gamma\lambda_1/(\gamma\lambda_1 + \lambda_2 - \gamma\lambda_2)]^{1/2}$$
$$\cdot I_1\{[4\gamma\lambda_1 t^2(\gamma\lambda_1 + \lambda_2 - \gamma\lambda_2)]^{1/2}\} \quad (A12)$$

[*Irving and Mullineux*, 1959], where we first used the substitution $t_c = t - t_f$ to eliminate the variable t_c and then eliminated t_f by de-

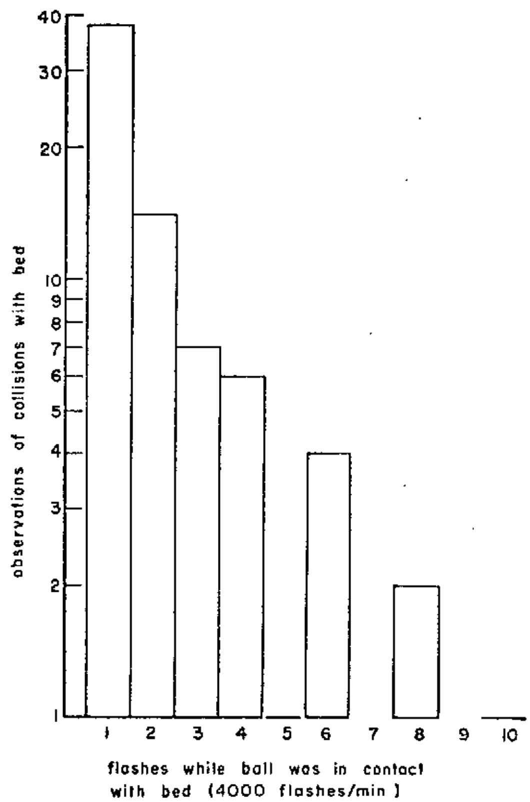

Fig. 10. State 2 distribution of rolling times for a ball in approximate contact with the bed and traveling at a reduced speed.

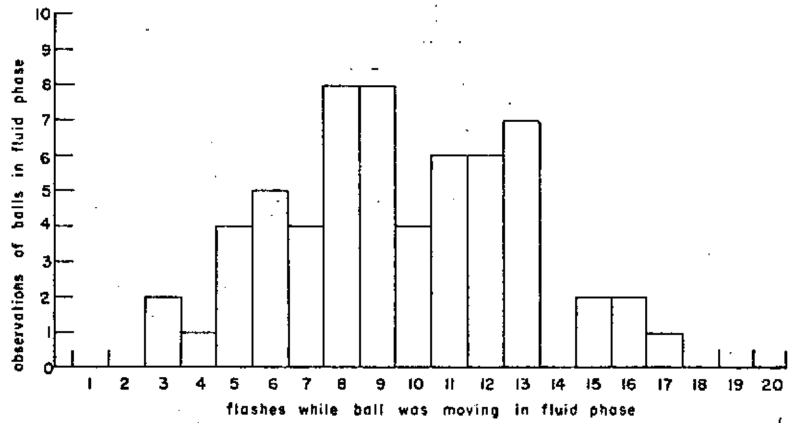

Fig. 11. State 1 distribution of times in the fluid phase between contacts (times spent in saltation arcs).

fining γ as the fraction of time spent in the fluid state by the sand grain while it traveled the prescribed distance; that is,

$$\gamma = t_f/t \qquad 0 < \gamma \leq 1 \qquad \text{(A13)}$$

I_1 is a first order Bessel function.

Consider now the following special simplification of the physical model. Assume that the times spent by the sand grain in the mobile phase are not Poisson distributed but constant. Let us assume further that this constant time is in fact the time required for the water to travel the prescribed distance (to be denoted as t_0).

If the random variable T_m associated with the times spent by the sand grain in the mobile phase can take on only one value (namely t_0), then equation A4*a* becomes

$$\begin{aligned} P_i(t_m) &= 1 \qquad t_m = t_0 \\ P_i(t_m) &= 0 \qquad \text{otherwise} \end{aligned} \qquad \text{(A14)}$$

McQuarrie [1963] has obtained the density function in time for this case by using the same general procedure described in the present work except that he solved equation A5*b* by using the simplification for $P_i(t_m)$ from equation A14 by Laplace transforms. From that point on our method follows step for step that of McQuarrie. He obtained the following expression for $P(t)$ by using equation A14:

$$P(t) = \exp(-\lambda_2 t - \lambda_1 t_0)\cdot(\lambda_1\lambda_2 t_0/t)^{1/2} \cdot I_1[(4\lambda_1\lambda_2 t t_0)^{1/2}] \qquad \text{(A15)}$$

$$x = t_f w = t_0 w \qquad \text{(A16)}$$

under the restriction of equation A14.

We define $\lambda_1/w = \lambda_1'$ and substitute the results for x in terms of t_0 from equation A16 into the expression for $P(t)$ in equation A15. The expression for sand transport originally derived by *Einstein* [1937] and rederived in a mysterious (to the present authors) fashion by *Todorović et al.* [1967] is then recovered. (Subscripts 1 and 2 must be switched.) We change the notation from $P(t)$ to $P(x, t)$ to indicate that the new probability distribution has been transformed by using equation A16 into a function of both distance and time:

$$P(x, t) = \exp(-\lambda_2 t - \lambda_1' x)\cdot(\lambda_1'\lambda_2 x/t)^{1/2} \cdot I_1[(4\lambda_1'\lambda_2 t x)^{1/2}] \qquad \text{(A17)}$$

Einstein's [1937] notation differs slightly from ours. His analysis proceeded from the a priori assumption of a proper set of units for time and distance so that the two rate constants λ_1' and λ_2 would be unity. In other words, he worked in reduced variables. To convert his expression to our equation A17, one need only substitute $\lambda_1' x$ and $\lambda_2 t$ for his values of x and t, respectively.

Einstein's probability distribution in reduced variable notation can also be formally obtained by the process of randomizing the γ probability distribution [*Feller*, 1966]. The physics of the transport process becomes difficult to visualize, however, when this formal technique is used. A full stochastic analysis of a single saltating grain of sand would require splitting state 2 into two cases: (1) a full stop, as treated above, and (2) rolling on the bed at a speed significantly less than that during free saltation. The experimental distribution of such rolling times is illustrated in Figure 10 (cf. Figure 2). It is considerably different from the distribution of times in the fluid state (Figure 11).

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